数理解析研究所講究録
672巻 1988年 14-25

TOPICS ON GAUSSIAN RANDOM FIELDS

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§1. Introduction

This report deals mainly with the Lévy Brownian motion $\{X(A), A \in R^d\}$, which serves as a guiding model throughout the theory of Gaussian random fields. A new method of the study of the Brownian motion is illustrated in the following three steps.

I. We first restrict the parameter to a curve C in a class $\mathbf{C}$ of C^∞-curves in R^2 space, then we are given one parameter Gaussian process, $\{X(t)=X(A(t)); A(t)\in C\}$, when the parameter $A(t)$ goes along the curve C where t is taken to be the arc length along C. The canonical representation theory can be applied to see the structure of $X(t)$ such as the Markov property.

II. Since the way of dependency of the Brownian motion is our main interest, we study the conditional expectation $E[X(P)/C]$ ($= E[X(P)/X(t); A(t)\in C]$) and further discuss the behaviour of the kernel function of $E[X(P)/C]$, expressed as a linear functional of $X(t)$, $A(t)\in C$.

III. We see from II that the conditional expectation can be viewed as a random functional of the curve C and hence the variation of which should be studied. By observing its variation, when the curve C changes, we might be able to see the way of dependency more clearly. Having been motivated by the expression of the variation $\delta E[X(P)/C]$, we discuss the normal derivative of X to observe the singularity of particular type.

To see the profound characteristic properties of Lévy's Brownian motion, we compare it with other typical Gaussian random fields, namely Wiener process and Ornstein Uhlenbeck process, from our view points.

§ 2. Canonical representation and conditional expectation

In this section we discuss the canonical representation of the Lévy Brownian motion on a smooth one dimensional manifold C in R^2 which is taken to be a simple curve of C^3-class, passing through the origin and then obtain the conditional expectation when the Lévy Brownian motion is given on the curve C. Denote the Brownian motion $X(A(t))$ at the point $A(t)$ on C by $X(t)$, t being taken to be the arc length along C with $A(0) = 0$. Then we are given one-dimensional parameter Gaussian process $\{X(t)/ A(t) \in C\}$ with $X(0) = 0$. First we consider the particular case where C is a circle and so the process can be written as $\{X(\theta)/0 \leqq \theta < 2\pi\}$. The covariance function of $X(\theta)$ is expressible as

$$(2.1) \qquad \Gamma(\theta, \theta') = \sin(\theta/2)(1-\cos(\theta'/2))+(1+\cos(\theta/2))\sin(\theta'/2), \qquad \theta \geqq \theta',$$

which tells us that if there exists the canonical representation, the kernel function has to be a Goursat kernel of order 2. Then we can prove the following theorem by using the canonical representation theory (see [6]).

<u>Theorem</u> 2.1 The Brownian motion $X(\theta)$ on a circle is a double Markov Gaussian process and has the canonical representation

$$X(\theta)=\int_0^\theta \{\sin(\theta/2)(\operatorname{cosec}(\theta'/2)- \frac{\cot(\theta'/4)}{2} h(\theta'))+ \cos^2(\theta/4)h(\theta')\}dB(\theta')$$

where $h(\theta) = \{ 1 + (\theta/4)\tan(\theta/4)\}^{-1}$.

We then take a curve C, starting from the origin, in a class $\mathbb{C}$ of C^3 -curves in R^2 and obtain an infinitesimal equation

$$(2.2) \qquad dX(t) = dB(t) + dt \int_0^t g(t,u)dX(u) + o(dt)$$

by using the fact that a part of C can be locally approximated by an arc of the osculation circle since the curvature of C is locally bounded. The following theorem can be proved by using (2.2).

Theorem 2.2 The Brownian motion on $C \in \mathbb{C}$ has the canonical representation

$$X(\tau) = \int_0^{\tau} F(\tau,u)dB(u) \tag{2.3}$$

where

$$F(\tau,u) = 1 + \int_u^{\tau} g(t,u)dt + \int_u^{\tau} dt \int_u^{t} du_1 g(t,u_1)g(u_1,u) + \dots \tag{2.4}$$

in which g is the solution of the Fredholm integral equation

$$\gamma_t(s) = \int_0^t \gamma(s,u)g_t(u)du - g_t(s), \qquad 0 < s < t, \tag{2.5}$$

with $g_t(u)$ and $\gamma_t(s)$ denoting $g(t,u)$ and $\gamma(t,s)$, respectively for fixed t. The kernel function $\gamma(s,u)$ in the expression (2.5) is symmetric and can be determined by

$$\frac{\partial^2}{\partial u\,\partial s}\Gamma(s,u) = \gamma(s,u) - \delta(s-u),$$

where $\Gamma(s,u)$ is the covariance of $X(s)$ and $X(u)$, and where δ is the delta function.

The conditional expectation $E[X(P)/C]$ can be written as a linear functional of $X(s)$

$$E[X(P)/C] = \int_C f_C(P,s)X(s)ds \tag{2.6}$$

in which f_C may involve some generalized functions. However, it can also be expressed in the form

$$E[X(P)/C] = \int_C g_C(P,s)\dot{B}(s)ds \tag{2.7}$$

with an L^2-kernel function g_C in terms of the white noise $\dot{B}(s)$ which comes from the canonical representation of $X(t)$. We give explicit expressions for some simple cases in the following examples.

Example 2.1 Let C be a line segment [0,a]. Then the kernel function f_C, in the expression (2.6), can be decomposed into two parts in the form,

$$f_C = f_1 + f_2 \quad ,$$

where

$$f_1 \propto \rho^{-3} \quad \text{and} \quad f_2 = c_a \delta_a \quad ;$$

where ρ denotes the distance between the point P and a general point A(s) on C, δ_a being the delta function at the boundary point a and c_a is a constant depending on a and P. However, if we change to the second form, the kernel function g_C is obtained as an L^2-function,

$$g_C(P,u) = \int_u^a f_1(s)ds + c_a.$$

Example 2.2 (A. Noda) Let C_a be a half line $\{(x,1);\ x \geqq a\}$, not passing through the origin 0. Then the Lévy Brownian motion restricted to C is a double Markov process taking $X(0) = 0$ and we can compute the conditional expectation

$$E[X(0,2)/X(x,1);\ x \geqq a] = \int_{C_a} f(s)X(s,1)ds$$

in which $f(s)$ is obtained as

$$f(s) = b(s^2 + 1)^{-3/2} + c_a \delta_a \ ,$$

where b and c_a are constants depending upon a.

Example 2.3 Let C be a circle segment $[\alpha,\beta]$, $\alpha \leqq 0 < \beta < 2\pi$. Then the conditional expectation E[X(P)/C] can be written in terms of Z(s) which is a strictly double Markov process such that $\mathcal{B}(X) = \mathcal{B}(Z)$ (see [6]) as follows:

$$E[X(P)/C] = \int_C f_C(P,s)Z(s)ds \tag{2.8}$$

where

$$f_C = f_{11} + f_{12} + f_2,$$

in which

$$f_{11} \propto \rho^{-3},$$

ρ being the distance between P and a point θ on C and f_{12} is indepndent of ρ and where

$$f_2 = c_{\alpha 1}\delta_\alpha + c_{\beta 1}\delta_\beta + c_{\alpha 2}\delta'_\alpha + c_{\beta 2}\delta'_\beta,$$

$c_{\alpha i}$ and $c_{\beta i}$ being constants. As in the above example, we can also find the kernel g_C which is an L^2-function.

Remark In Example 2.1, if "a" tends to infinity then f_2 (δ-function at the boundary point "a") will disappear. Similarly in Example 2.3, if the curve is closed, i.e. a circle ($\alpha=0$, $\beta=2\pi$), we do not need δ-functions (see [6], . §3, Example 2).

§3. Variation of the conditional expectation

As was discussed in the previous section the conditional expectation E[X(P)/C] can be viewed as a functional of the curve C. We are now interested in its variation when the curve C varies in a class $\mathbf{C}$ of C^∞-curves. The variational calculus of random functionals (of a curve C) has not been much discussed yet, however, the S-transform, in Hida's calculus ([3],[4]), carries them to ordinary functionals on a function space, so that we can appeal to the classical variation theory of ordinary functionals.

We know that the Lévy Brownian motion X(A) can be expressed by

$$(3.1) \qquad X(A) = c(d) \int_{S(A)} |u|^{-(d-1)/2}\, W(u)du; \quad u \in R^d ,$$

in terms of white noise W , where

$$(3.2) \qquad S(A) = \{u;\ (u,OA) \geqq |u|^2\}$$

and

$$(3.3) \qquad c(d) = \{ \frac{2(d-1)}{|S|^{d-1}} \int_0^{\pi/2} \sin^{d-2}\theta \ d\theta\}^{1/2}.$$

The S-transform of $X(A)$ is

$$(3.4) \qquad (SX(A))(\xi) = c(d)\int \chi_{S(A)}(u)|u|^{-(d-1)/2}\, \xi(u)du.$$

Now we can rewrite the higher dimensional version of the expression (2.7), for fixed P, in terms of white noise W and so the S-transform of $E[X(P)/C]$, denoted by $U(C,\xi)$, is obtained as follows.

$$(3.5) \qquad U(C,\xi) = \int_C ds f(C,s) \int_{R^d} g(s,u)\, \xi(u)du,$$

where

$$(3.6) \qquad g(s,u) = c(d)\, \chi_{S(A(s))}(u)\, |u|^{-(d-1)/2} .$$

By appealing to the Lévy variation formula (see [1]), the variation of $U(C,\xi)$ can be expressed by

$$(3.7) \qquad \begin{aligned} \delta U(C,\xi) &= \int ds\{\delta f(C,s) - \kappa f(C,s)\,\delta n(s)\}\int g(s,u)\,\xi(u)du \\ &+ \int ds f(C,s)\ \delta n(s) \int \frac{\partial}{\partial n}\, g(s,u)\,\xi(u)du, \end{aligned}$$

where $\partial/\partial n$ and $\delta n(s)$ denote the normal differential operator and the distance between C and $C+\delta C$ at the point $A(s)$ respectively, and where $\delta f(C,s)$ is the variation of the kernel f and κ is the curvature of a curve C (see [7]). This formula would guarantee the existence of the stochastic version $\delta E[X(P)/C]$ which is expressed in the form

$$\delta E[X(P)/C] = \int_C \{\delta f(C,s) - \kappa f(C,s)\delta n(s)\}X(s)ds + \int_C f(C,s)\frac{\partial}{\partial n}X(s)\delta n(s)ds. \tag{3.8}$$

The normal derivative $\partial X/\partial n$, appeared in this formula, suggests us to think of its probabilistic structure, and then, in the simplest case of a circle C, we obtain the following proposition.

<u>Proposition</u> 3.1 Let $\{X(\theta) \equiv X(A(\theta)),\ 0 \leqq \theta \leqq 2\pi\}$ be the Lévy Brownian motion on a circle $C\ (\subset R^2)$ with radius t. Then the normal derivative $\partial X/\partial n$ on C is neither an ordinary process nor a generalized process, however it is well defined as a generalized process over R^2.

This proposition can be easily extended to the cases of C being taken as (i) a straight line and (ii) a general curve in the class $\mathbf{C}$. In addition, we can prove a more general result which is described in the following theorem.

<u>Theorem</u> 3.1 Let $X(A)$, $A \in R^d$, be a Gaussian random field such that

$$E[X(A)] = 0,\ E[X(A)-X(B)]^2 = c\rho(A,B) + \phi(\rho(A,B)),\ c > 0, \tag{3.9}$$

where ρ is the Euclidean distance in R^d and where ϕ is a $C^2(0,\infty)$-function with $\phi(0) = \phi'(0) = 0$. Let S be a $(d-1)$-dimensional surface in R^d which is an anlytic manifold. Then

(i) for $d = 2$, $\partial X/\partial n\big|_S$ is not well defined, while $\partial X/\partial n$ is well defined as a generalized Gaussian random field over R^2,

(ii) for $d > 2$, $\partial^k X/\partial n^k\big|_S$, $k \leqq (d-1)/2$, exist as generalized Gaussian random fields over S where $\partial X/\partial n\big|_S$ denotes the restriction of $\partial X/\partial n$ to the manifold S.

The most interesting process satisfying (3.9) is the Lévy Brownian motion. Another interesting example is obtained by taking

$$\phi(t) = \int_0^\infty (1-e^{-\lambda t^2})d\sigma(\lambda) \quad \text{with} \quad \int_0^\infty \lambda \, d\sigma(\lambda) < \infty.$$

§4. Comparison with Wiener and Ornstein Uhlenbeck processes

We wish to compare Lévy's Brownian motion with other Gaussian random fields in aspect of the results obtained in the previous sections. For this purpose we are going to discuss Wiener process and Ornstein-Uhlenbeck process.

(1) Wiener process

Let $Y = \{Y(u);\ u = (u_1,\ldots,u_d) \in (R_+)^d\}$, $d \geqq 2$, be d-dimensional Wiener process. Namely, Y is a Gaussian random field such that

(4.1) $\qquad E[Y(u)] = 0,\ E[Y(u)Y(v)] = \prod_i (u_i \wedge v_i);$

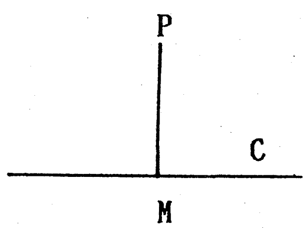
where $u = (u_1,\ldots,u_d)$ and $v = (v_1,\ldots,v_d)$.

Let P be a fixed point in R^2, and let C be a curve in the class $\mathbf{C}$, introduced in §1. Then, we are given one dimensional Gaussian process $\{Y(t) \equiv Y(A(t)),\ A(t) \in C\}$. The conditional expectation of Wiener process is discussed in the following examples.

Example 4.1 Let C be a line segment in R^2 which is parallel to coordinate axis. Then

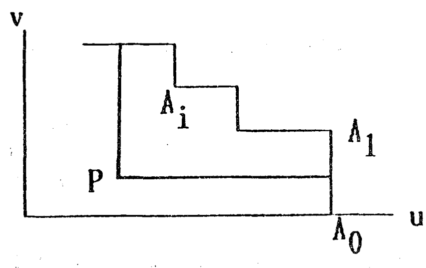
$$E[Y(P)/C] = Y(M),$$

where M is the base point of the normal line of C passing through the point P.



Example 4.2 Let C be $\bigcup_1^n C_i$, where each C_i is a line segment parallel to coordinate axis and assume that C is continuous monotone decreasing. For computation, we let the process starts from the point A_0 on the u-axis.

(4.2) $E[Y(P)/C] = \sum_1^n c_i\, Y(A_i) + \sum_1^k d_i Y(B_i)$,



where A_i's are the vertices of C and B_i, $i=1,..,k$, $k \leqq 2$, are the base points of the normal line of C passing through P, and where $c_i = f(P)/f(A_i)$ and $d_i = f(P)/f(B_i)$, in which $f(A)$ denotes the area of the rectangular with the vertices including the point A and the origin .

In a similar manner, we can discuss the case when C is increasing.

Comparing Example 2.1 with Example 4.1, we see a big difference between the Levy Brownian motion and Wiener process. Now take C as in Example 4.2. If the conditional expectation is expressed in the form (2.6), then it is observed that the kernel function for Wiener process is a very simple one consisting of only δ-functions but it would be complicated for Lévy's Brownian motion.

Proposition 4.1 Let C be given as in the above Example 4.2 . Then the Gaussian process $\{Y(t);\ A(t)\in C\}$ is simple Markov.

Concerning normal derivative we can prove the following proposition.

Proposition 4.2 (i) Let S be $\{u = (u_1,..,u_i,..,u_d) \in (R_+)^d;\ u_i = c\}$, for $1 \leqq i \leqq d$. Then the normal derivative $\dfrac{\partial Y(u)}{\partial u_i}$ on S is not well defined, however $\left.\dfrac{\partial Y(u)}{\partial u_k}\right|_S$ is well defined as a generalized Gaussian random field if $k \neq i$.

(ii) The normal derivative $\partial Y(u)/\partial n$ on a spherical surface $S^{d-1}(\subset (R_+)^d)$ is well defined as a generalized Gaussian random field for all $d \geqq 2$.

Remark For d = 2, $\left.\dfrac{\partial Y(u_1,u_2)}{\partial u_2}\right|_{u_1 = a}$ can be viewed as a white noise.

The normal derivative of the Wiener process on a hyperplane always has singularity. But the normal derivative of the Lévy Brownian motion on a hyperplane is well defined for $d > 2$.

(2) Ornstein Uhlenbeck process

Let $\{U_m(u);\ u \in R^d\}$, $d \geqq 2$ be Ornstein Uhlenbeck process with mass parameter $m > 0$. It is known to be a generalized Gaussian random field with characteristic functional

$$
\begin{aligned}
C(\xi) &= E[\exp\{i \int U_m(u)\,\xi(u)du\}] \\
&= \exp[-\frac{1}{2}\int_{R^d} \frac{\hat{\xi}(\lambda)^2}{m^2+\lambda^2}\,d\lambda], \qquad (4.3)
\end{aligned}
$$

where $\hat{\xi}$ denotes the Fourier transform of ξ.

Proposition 4.3. The normal derivative $\partial U_m / \partial n$ is again a generalized Gaussian random field on the whole space R^d, but it is not well defined when we restrict the parameter to a hyperplane.

The normal derivative $\partial U_m / \partial n \,\big|_{R^{d-1}}$ is, as it were, a limit of fields expressed as a superposition of mutually independent (d-1)-dimensional known random fields as follows:

$$
(4.4) \qquad \partial U_m / \partial n \,\Big|_{R^{d-1}} = \lim_{N \to \infty} \int_0^N \lambda\, U_{a(m,\lambda)}\, d\lambda,
$$

where $a(m,\lambda) = \sqrt{m^2 + \lambda^2}$. It can be easily seen from its characteristic functional

$$(4.5)\qquad C(\xi_0) = \exp[-\frac{1}{2}\int_{R^1} d\lambda_d \ \lambda_d^2 \int_{R^{d-1}} \frac{\xi_0(\lambda_1, \ldots, \lambda_{d-1})}{(m^2+\lambda_d^2)+\lambda_1^2+\ldots+\lambda_{d-1}^2} d\lambda \ .$$

We now take S to be a circle with radius 1. Consider Ornstein Uhlenbeck process $U_m(r,\theta)$ in R^2-space. Then the restriction of the normal derivative of R^2-parameter Ornstein Uhlenbeck process to S is denoted by $\left.\frac{\partial U_m(r,\theta)}{\partial n}\right|_{r=1}$. Its characteristic functional is obtained as

$$(4.6)\qquad C(\tilde{\xi}) = \exp\{-\frac{1}{2}\int d\rho \int \frac{\rho^3}{m^2+\rho^2} \gamma_\rho(\theta, \theta') \tilde{\xi}(\theta) \tilde{\xi}(\theta') d\theta \, d\theta'\},$$

where

$$\gamma_\rho(\theta, \theta') = \int_0^{2\pi} \cos(\theta-\phi)\cos(\theta'-\phi)\, e^{i\rho(\cos(\theta-\phi) - \cos(\theta'-\phi))}\, d\phi$$

and where $\tilde{\xi}(\theta)$ is a factor of the original test function $\xi(r,\theta)$ which is taken as $g(r)\tilde{\xi}(\theta)$.

Since $\gamma_\rho(\theta, \theta)$ is positive definite, we can write $\left.\frac{\partial U_m(r,\theta)}{\partial n}\right|_{r=1}$ as follows:

$$(4.8)\qquad \partial U_m / \partial n \Big|_S = \lim_{N\to\infty} \int_0^N \sqrt{\frac{\rho^3}{m^2+\rho^2}}\ Y_\rho(\theta)\, d\rho$$

where $Y_\rho(\theta)$ is a Gaussian process with covariance function $\gamma_\rho(\theta, \theta)$.

<u>Remark</u> The expression (4.8) is obtained by Professor A. Noda's comment.

It is noted that the singularity of the normal derivative always occurs in R^d-parameter case for every $d \geqq 2$.

<u>Acknowledgement</u> The author is grateful to Professor A. Noda for his help and suggestion for generalization.

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